CÁLCULO 3: FUNCIONES DE EULER

La función Gamma

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx$$

Ejemplos

$\Gamma(3) = \int_0^\infty x^2 e^{-x}\,dx = 2!$

$\Gamma(1/2) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \sqrt{\pi}$

$$\Gamma(4/3) = \int_0^\infty \sqrt[3]{x}\,e^{-x}dx$$

Si p$<$ 0 , la integral es infinita.

Demostración:

$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx = \int_0^1 x^{p-1}e^{-x}\,dx + \int_1^\infty x^{p-1}e^{-x}\,dx$

$e^{-1}x^{p-1} \leq x^{p-1}e^{-x} \leq x^{p-1}$

$\int_0^1 x^{p-1}e^{-x}\,dx = \int_0^1 \frac{1}{x^{1-p}}\,dx$ converge si p > 0

$\lim_{x\to\infty} x^{p-1}e^{-x} = 0$, $0 < x^{p-1}e^{-x} \leq k < \infty$

$\int_1^\infty x^{p-1}e^{-x}\,dx \leq k\int_1^\infty \frac{1}{x^2}\,dx < \infty$

Propiedades de la función Gamma

1. $\Gamma(p) = (p-1)\cdot\Gamma(p-1) \quad si\ p > 1$

   $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx = \binom{u=x^{p-1}}{dv=e^{-x}} = [x^{p-1}(-e^{-x})]_0^\infty -$
   $$\int_0^\infty -e^{-x}(p-1)x^{p-2}\,dx = (p-1)\int_0^\infty x^{p-2}e^{-x}\,dx = (p-1)\Gamma(p-1)$$

2. Si p= m ∈ $\mathbb{N}$, , $\Gamma(m) = (m-1)!$

$\int_0^{\infty} x^0 e^{-x}\, dx = [(-e^{-x})]_0^{\infty} = (-0) - 1 = 1$

3. Si p= m+ r siendo r la parte decimal ,,
   $\Gamma(m+r) = (m+r-1)(m+r-2) \ldots (1+r) \cdot r \cdot \Gamma(\mathrm{r})$

   $\Gamma(3{,}2) = 2{,}2 \cdot 1{,}2 \cdot 0{,}2 \cdot \Gamma(0{,}2)$

4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Ejemplos

$$\Gamma(5) = 4 \cdot \Gamma(3) = 4 \cdot \int_0^{\infty} \mathrm{x}^2\, \mathrm{e}^{-\mathrm{x}} \mathrm{dx} = 4!$$

$$\Gamma\left(\frac{7}{3}\right) = \frac{4}{3} \cdot \frac{1}{3} \cdot \Gamma\left(\frac{1}{3}\right) = \frac{4}{9} \cdot \Gamma\left(\frac{1}{3}\right)$$

$$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{105\sqrt{\pi}}{16}$$

5. $\Gamma(\mathrm{p}) \cdot \Gamma(1-\mathrm{p}) = \frac{\pi}{\sin \mathrm{p}\pi}$, $0 < p < 1$

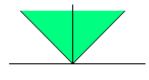
Ejemplos

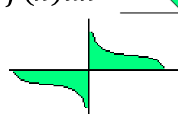
$$\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin \pi/3}$$

$$\int_0^{\infty} \sqrt{x}\, e^{-x^3}\, dx = \begin{pmatrix} y = x^3\,, \quad x = y^{\frac{1}{3}} \\ dx = \frac{1}{3} y^{-\frac{2}{3}} dy \end{pmatrix} = \int_0^{\infty} y^{\frac{1}{3}\frac{1}{2}} \cdot e^{-y} \cdot \frac{1}{3} y^{-\frac{2}{3}}\, dy =$$

$$= 1/3 \int_0^{\infty} y^{-1/2} \cdot e^{-y}\, dy = \sqrt{\pi}\,/3$$

Integrales según si la función es par o impar:

Si f es par, $\int_{-a}^{a} f(x)dx = 2 \cdot \int_0^{a} f(x)dx$ 

Si f es impar, $\int_{-a}^{a} f(x)dx = 0$ 

La función Beta

$$\beta(p,q) = \int_0^1 x^{p-1}\,(1-x)^{q-1}dx \qquad p,q > 0$$

β es impropia si p o q son menores que 1.

Propiedades de la función Beta

1. $\beta(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$

2. $\beta(p,q) = 2\cdot \int_0^{\frac{\pi}{2}} sin^{2p-1}\theta \cdot cos^{2q-1}\theta\; d\theta$

   Ejemplo

   $\beta(p,q) = \int_0^1 x^{p-1}\cdot(1-x)^{q-1}\,dx = \int_0^{\frac{\pi}{2}} sin^{2p-2}\theta\cdot cos^{2q-2}\cdot 2sin\theta cos\theta\; d\theta =$

   $\left(\begin{matrix} x = sin^2\theta \;\; ; \;\; 1-x = cos^2\theta \\ dx = 2sin\theta cos\theta d\theta \end{matrix}\right)$ *hacer la sustitución*

3. $I(n,m) = \int_0^{\frac{\pi}{2}} sin^n\theta\cdot cos^m\theta\; d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$

   Ejemplos

   $$\int_0^{\frac{\pi}{2}} sin^7\theta\cdot cos^5\theta\; d\theta = \frac{1}{2}\beta(4,3) = \frac{1}{2}\cdot\frac{\Gamma(4)\cdot\Gamma(3)}{\Gamma(7)} = \frac{1}{2}\cdot\frac{3!\,2!}{6!} = \frac{1}{6\cdot5\cdot4} = \frac{1}{120}$$

   $$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{sinxcosx}}dx = \int_0^{\frac{\pi}{2}} (sinx)^{-1/2}\cdot(cosx)^{-1/2}\,dx = \frac{1}{2}\cdot\,\beta\left(\frac{1}{4},\frac{1}{4}\right) = \frac{1}{2}\cdot\frac{(\Gamma(1/4))^2}{\Gamma\left(\frac{1}{2}\right)}$$

EJERCICIOS

$\int_0^1 \sqrt{(x-x^2}dx = \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx = \beta\left(\frac{3}{2},\frac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} = \frac{\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2!} = \frac{\pi}{8}$

$$\int_0^\infty xe^{-x^5}dx = \begin{pmatrix} u = x^5 & x = u^{\frac{1}{5}} \\ dx = \frac{1}{5}\cdot u^{-\frac{4}{5}}du \end{pmatrix} = \int_0^\infty u^{\frac{1}{5}}e^{-u}\cdot\frac{1}{5}u^{-\frac{4}{5}}du =$$

$1/5\int_0^\infty u^{-\frac{3}{5}}e^{-u}du = \frac{\Gamma\left(\frac{2}{5}\right)}{5}$

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}dx = \int_0^1 (1-x^4)^{-\frac{1}{2}}dx = \begin{pmatrix} u = x^4 & x = u^{\frac{1}{4}} \\ dx = \frac{1}{4}u^{-\frac{3}{4}}du \end{pmatrix}$$

$$= \int_0^1 (1-u)^{-\frac{1}{2}}\cdot\frac{1}{4}u^{-\frac{3}{4}}du = \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\frac{3}{4})} = \frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$= \frac{\sqrt{\pi}}{4\pi\sqrt{2}}\Gamma\left(\frac{1}{4}\right)^2 = \frac{\sqrt{8\pi}}{8\pi}\Gamma\left(\frac{1}{4}\right)^2$

$$\int_0^{\frac{\pi}{2}} \sqrt{tanx}\,dx = \int_0^{\frac{\pi}{2}} (sinx)^{\frac{1}{2}}(cosx)^{-\frac{1}{2}}dx = \beta\left(\frac{3}{4},\frac{1}{4}\right) = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \pi\sqrt{2}$$

$\int_{-\infty}^{\infty} x^2e^{-n^2x^2}dx = 2\int_0^\infty x^2e^{-nx^2}dx \to si\ n \neq 0$

$\to \begin{pmatrix} u = n^2x^2 & x = \frac{1}{n}u^{1/2} \\ dx = \frac{1}{2n}u^{-\frac{1}{2}}du \end{pmatrix} = 2\int_0^\infty \frac{1}{n^2}\cdot ue^{-u}\cdot\frac{1}{2n}u^{-1/2}du =$

$\frac{1}{n^3}\cdot\int_0^\infty u^{1/2}e^{-u}du = \frac{1}{n^3}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{1}{n^3}\frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2n^3}$

$$\int_0^1 x^m(\ln\frac{1}{x})^p\,dx = \begin{pmatrix} x = e^{-u} & \frac{1}{x} = e^u \\ \ln\frac{1}{x} = u & dx = e^{-u}du \end{pmatrix}\begin{matrix} x = 0 \to u = \infty \\ x = 1 \to u = 0 \end{matrix} =$$

$$\int_\infty^0 e^{-mu}u^p - e^{-u}du \to \left[\int_a^b f = -\int_b^a f\,]\right] \to \int_0^\infty u^pe^{-(m+1)u}du =$$

$$\begin{pmatrix} y = (m+1)u \\ dy = (m+1)du \end{pmatrix} = \int_0^\infty \frac{y^p}{(m+1)^p}e^{-y}\frac{dy}{m+1} = \frac{\Gamma(p+1)}{(m+1)^{p+1}}$$

Converge si p+1>0